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THEORETICAL ASPECTS OF DROPLET IMPACT MEASUREMENTS

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The Pioneering Research Laboratory on Physics of Fine Particles was established in 1962, at the Ohio Agricultural Experiment Station, Wooster, by the Agricultural Research Service of the United States Department of Agriculture, in recognition of the need for continued fundamental research on fine particle behavior and, in particular, the criticality of such studies to agriculture.

In agriculture, pesticides in both liquid droplet and solid particle form are used extensively, but their use is frequently hampered by lack of optimum efficiency and precision of application, despite the best efforts of manufacturers and applicators. This results in an excessive economic burden on agriculture in the maintenance of the best of quality in agricultural produce, as well as an increased chance of contaminating surrounding areas. Problems of air pollution are also of prime pertinence and concern. It is desired that the results of these investigations in fine particle behavior and its inherent subject matter will serve agriculture and its allied industries, and other industries with similar vexations, in the alleviation of these problems.

THEORETICAL ASPECTS OF DROPLET IMPACT MEASUREMENTS

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The damped spring-mass transducer is considered from the standpoint of droplet impaction measurements. The impulse concept of droplet impaction is mentioned. Responses of the transducer to step-function and random inputs are analyzed theoretically.

INTRODUCTION

The linear spring-mass system, in the form of a cantilever beam, has found application to the study of liquid droplet impaction.[2] In this paper, we will consider some theoretical aspects of the measurement technique. It will not affect the discussion to assume that the mass of the system remains constant, i.e., no liquid is accumulated by the impaction surface.

DROPLET MOMENTUM

The major force causing the deflection of the spring-mass system will arise from the summation of the individual droplet impulses. The mass of liquid in each droplet will contribute an impulse dependent upon its momentum change in the direction of the transducer deflection. This follows from the definition of the impulse, or time integral of a force $\mathbf{F}$,

$$\int_{t_0}^{t} \mathbf{F}\,dt = \int_{t_0}^{t} m\,\frac{d^2\mathbf{r}}{dt^2}\,dt = m\left(\frac{d\mathbf{r}}{dt}\right)_{t=t} - m\left(\frac{d\mathbf{r}}{dt}\right)_{t=t_0} = \mathbf{u} - \mathbf{u}_0 \tag{1}$$

where m is the mass, $\mathbf{r}$ is the position vector, and $\mathbf{u}_0$ and $\mathbf{u}$ are the momentum vectors at $t=t_0$ and $t=t$, respectively.

We would expect that the energy involved in the momentum change would be spent in shattering of the droplet, viscous dissipation in the fluid and surrounding air, and transfer of momentum to directions other than the line of impact. Also, turbulence should result in the residual fluid and heat should appear through various processes. A target such as soil would take up energy through deformation and breakup.

RESPONSE OF THE MEASUREMENT TRANSDUCER TO A STEP FUNCTION INPUT

The behavior of the damped spring-mass transducer may be described by the differential equation

$$m\,\frac{d^2x}{dt^2} + c\,\frac{dx}{dt} + kx = F(t), \tag{2}$$

where m and k are the apparent mass and spring constant, respectively, c is the damping coefficient, and

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[2] P. E. Schleusener and E. H. Kidder. Agr. Engineering 41: 100–3. 1960.

$F(t)$ is the impressed force. We apply a force $F(t)$ in the form of a unit step function, i.e.,

$$F(t) = 1(t - t_0). \tag{3}$$

It is convenient to define, in the usual manner, the undamped natural angular frequency,

$$\omega_n = (k/m)^{1/2}; \tag{4}$$

the critical damping coefficient,

$$c_c = 2m\omega_n; \tag{5}$$

and the damping factor

$$\zeta = c/c_c. \tag{6}$$

Upon obtaining the general solution for (2) under these definitions and the initial conditions

$$x(0+) = x_0 = 0; \tag{7}$$

$$\dot{x}(0+) = \dot{x}_0 = 0, \tag{8}$$

it is possible to distinguish among the cases $\zeta < 1$, $\zeta = 1$, and $\zeta > 1$.

Case I. $\zeta < 1$ (Light Damping). For $\zeta < 1$, the general solution becomes

$$x(t) = \frac{1(t-t_0)}{m\omega_n^2}\left\{1 - e^{-\zeta\omega_n(t-t_0)}\left[\cos(\omega_n(1-\zeta^2)^{1/2}(t-t_0)) + \frac{\zeta}{(1-\zeta^2)^{1/2}}\sin(\omega_n(1-\zeta^2)^{1/2}(t-t_0))\right]\right\} \tag{9}$$

Hence, the displacement $x(t)$ of the transducer in response to a step function input may be represented by a damped oscillation superimposed upon a step function.

Case II. $\zeta = 1$ (Critical Damping). If $\zeta = 1$, the application of a limiting process and L'Hospital's rule to the general solution yields the result

$$x(t) = \frac{1(t-t_0)}{m\omega_n^2}\left\{1 - [1 + \omega_n(t-t_0)]e^{-\omega_n(t-t_0)}\right\}. \tag{10}$$

This case represents the transition from the oscillatory to the nonoscillatory condition of the transducer. An aperiodic displacement is superimposed upon the step function.

Case III. $\zeta > 1$ (Heavy Damping). In this case, the general solution takes the form

$$x(t) = \frac{1(t-t_0)}{m\omega_n^2}\left\{1 - e^{-\zeta\omega_n(t-t_0)}\left[\cosh(\omega_n(\zeta^2-1)^{1/2}(t-t_0)) + \frac{\zeta}{(\zeta^2-1)^{1/2}}\sinh(\omega_n(\zeta^2-1)^{1/2}(t-t_0))\right]\right\} \tag{11}$$

An aperiodic curve is superimposed upon the step function displacement.

Obviously, the lightly damped transducer may indicate oscillations which are not representative of the input. This would be particularly true for a sudden change in the input, such as indicated by the above results for the step-function input. Accordingly, it would appear desirable to damp the transducer in the critical to heavy range.

The transducer will act as a band-pass filter with characteristics dependent upon the natural frequency and damping. Therefore, it is now pertinent to describe its response to a random input.

RESPONSE OF THE TRANSDUCER TO A RANDOM FORCE

If the transducer is subjected to a random force $U(t)$, the displacement becomes a random process governed by the stochastic differential equation

$$\ddot{X}(t) + 2\omega_n\zeta\dot{X}(t) + \omega_n^2 X(t) = U(t). \tag{12}$$

The symbols $\ddot{X}(t)$ and $\dot{X}(t)$ denote the random acceleration and velocity, respectively. The processes $X(t)$ and $U(t)$ are assumed stationary and allowed to be possibly complex with the expectations

$$E\{X(t)\} = O;\ E\{U(t)\} = O, \tag{13}$$

for convenience, and

$$E\{X(t)X^*(t)\} = \sigma_x{}^2;\ E\{U(t)U^*(t)\} = \sigma_u{}^2. \tag{14}$$

Since $X(t)$ and $U(t)$ are stationary, they may be analyzed into orthogonal components in the manner suggested by Bartlett (*p. 168*)[3] or Batchelor (*p. 55*)[4]. The generalized harmonic analyses of $X(t)$ and $U(t)$ in terms of the orthogonal processes $Z(\omega)$ and $W(\omega)$ will be

$$X(t) = \int_{-\infty}^{\infty} e^{it\omega} dZ(\omega), \tag{15}$$

and

$$U(t) = \int_{-\infty}^{\infty} e^{it\omega} dW(\omega). \tag{16}$$

Then (12) becomes

$$\int_{-\infty}^{\infty} e^{it\omega}(-\omega^2 - 2i\omega_n\zeta\omega + \omega_n^2)dZ(\omega) = \int_{-\infty}^{\infty} e^{it\omega} dW(\omega), \tag{17}$$

which upon combination with its complex conjugate form gives

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[(\omega_n^2 - \omega^2)^2 + (2\omega_n\zeta)^2\omega^2]\,dZ(\omega)dZ^*(\omega) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dW(\omega)\,dW^*(\omega). \tag{18}$$

Taking the expectation of both sides of (18) and then interchanging the operations of averaging and integration, we obtain

$$\int_{-\infty}^{\infty}[(\omega_n^2 - \omega^2)^2 + (2\omega_n\zeta)^2\omega^2]\,dE\{Z(\omega)Z^*(\omega)\} = \int_{-\infty}^{\infty} dE\{W(\omega)W^*(\omega)\}. \tag{19}$$

It is now helpful to define

$$Y(\omega) = E\{|Z(\omega)|^2\} \tag{20}$$

and

$$V(\omega) = E\{|W(\omega)|^2\} \tag{21}$$

[3] M. S. Bartlett. *An Introduction to Stochastic Processes.* 312 pp. London, 1956.
[4] G. K. Batchelor. *The Theory of Homogeneous Turbulence.* (Cambridge University Press, London, 1953.)

where $Y(\omega)$ and $V(\omega)$ are never-decreasing functions such that

$$E\{X(t)X^*(t)\}=E\{|X(t)|^2\}=\sigma_x{}^2=\int_{-\infty}^{\infty}dE\{Z(\omega)Z^*(\omega)\}=\int_{-\infty}^{\infty}dY(\omega)=Y(\infty)-Y(-\infty), \tag{22}$$

and similarly

$$E\{|U(t)|^2\}=\sigma_U{}^2=\int_{-\infty}^{\infty}dV(\omega)=V(\infty)-V(-\infty), \tag{23}$$

with

$$Y(-\infty)=O; V(-\infty)=O, \tag{24}$$

and

$$Y(\infty)=\sigma_x{}^2; V(\infty)=\sigma_U{}^2. \tag{25}$$

We now make the reasonable assumption that $X(t)$ and $U(t)$ have continuous spectra, with spectral functions $F_x(\omega)$ and $F_U(\omega)$ and spectral densities $f_x(\omega)$ and $f_U(\omega)$ for which

$$dF_x(\omega)=f_x(\omega)d\omega; \tag{26}$$

$$dF_U(\omega)=f_U(\omega)d\omega. \tag{27}$$

Then it follows that

$$E\{X^*(t)X(t+\tau)\}=w_x(\tau)=\sigma_x{}^2\rho_x(\tau)=\sigma_x{}^2\int_{-\infty}^{\infty}e^{i\tau\omega}dF_x(\omega)=\sigma_x{}^2\int_{-\infty}^{\infty}e^{i\tau\omega}f_x(\omega)d\omega=\int_{-\infty}^{\infty}e^{i\tau\omega}dY(\omega), \tag{28}$$

where $w_x(\tau)$ and $\rho_x(\tau)$ are the autocovariance and autocorrelation functions, respectively. A similar procedure for $U(t)$ yields

$$E\{U^*(t)U(t+\tau)\}=w_U(\tau)=\sigma_U{}^2\rho_U(\tau)=\sigma_U{}^2\int_{-\infty}^{\infty}e^{i\tau\omega}f_U(\omega)d\omega=\int_{-\infty}^{\infty}e^{i\tau\omega}dV(\omega). \tag{29}$$

Therefore, on the bases of (28) and (29),

$$dY(\omega)=\sigma_x{}^2f_x(\omega)d\omega; \tag{30}$$

$$dV(\omega)=\sigma_U{}^2f_U(\omega)d\omega. \tag{31}$$

Returning to (19), we now find it possible to write

$$\int_{-\infty}^{\infty}[(\omega_n^2-\omega^2)^2+(2\omega_n\zeta)^2\omega^2]dY(\omega)=\int_{-\infty}^{\infty}dV(\omega), \tag{32}$$

and consequently,

$$f_x(\omega)=\frac{\sigma_U{}^2f_U(\omega)}{\sigma_x{}^2[(\omega_n^2-\omega^2)^2+(2\omega_n\zeta)^2\omega^2]}, \tag{33}$$

which is the relationship between the input and output spectral densities.

It is of interest to consider also the effect of a general linear transducer or operator upon the input. Here we simply follow the method outlined by Bartlett (*pp. 174–175*)[5] by defining the linear operator

$$X(t)=L\{U(t)\}=\int_{-\infty}^{\infty}g(t-u)U(u)du. \tag{34}$$

[5] See footnote 3.

In the theory of generalized harmonic analysis (Bartlett, *p. 169*), it is shown that (34) is equivalent to

$$X(t)=\int_{-\infty}^{\infty} e^{it\omega}h(\omega)\,dW(\omega), \tag{35}$$

where $W(\omega)$ is an orthogonal process as in (16), and

$$h(\omega)=\int_{-\infty}^{\infty} e^{-iv\omega}g(v)\,dv \tag{36}$$

For the autocovariance of $X(t)$ we have

$$E\{X^*(t)X(t+\tau)\}=\int_{-\infty}^{\infty} e^{i\tau\omega}h(\omega)h^*(\omega)\,dV(\omega), \tag{37}$$

so that by virtue of (31),

$$\sigma_x^2 f_x(\omega)=h(\omega)h^*(\omega)\sigma_U^2 f_U(\omega). \tag{38}$$

A particular case of the linear operator $L\{\ldots\}$ is the linear differential operator $\varnothing(d/dt)=\varnothing(D)$. Eq. (38) gives formally, then, for a $\varnothing(i\omega)$ which is non-vanishing for real ω

$$\sigma_x^2 f_x(\omega)=\frac{\sigma_U^2 f_U(\omega)}{|\varnothing(i\omega)|^2}, \tag{39}$$

corresponding to the inverse linear operator L^{-1}. For the damped oscillator transducer,

$$\varnothing(D)=D^2+\lambda D+\omega_n^2, \tag{40}$$

so that the result given by (33) follows immediately. Hence, it is seen that the operational formalism facilitates the consideration of any linear transducer for its output as a result of a given input.

If the spectral characteristics of the output are determined, then one should be able to estimate the unknown input spectrum by means of (33). Of course, an alternate procedure would be to make autocorrelation measurements as indicated by (28), since the spectral and autocorrelation functions are Fourier-transform related.

From (33), it is noted that for light damping, the damped oscillator transducer will have a response sharply tuned to its natural frequency. As mentioned in the preceding section, in this state the transducer will act as a band-pass filter. It might in this sense serve as a part of a spectrum analysis system, but not necessarily a convenient one. It would further, give the false indications of oscillations on ordinary recorder traces. A good degree of damping would certainly be necessary for any averaging force measurements. Accordingly, the value of this particular transducer for more detailed studies of the droplet impaction process may be limited, or at least subject to question.

IMPLICATIONS FOR THE IMPACTION PROCESS

One may speculate about the physical factors which would be reflected in droplet impaction spectra. It would seem reasonable to expect that among these factors would be found:

(1) the manner of breakup and dissipation of individual droplet momenta;
(2) droplet size spectra;
(3) delivery characteristics of the droplet generation system;
(4) characteristic turbulence of the ambient medium;
(5) effects of ambient medium turbulence induced by the droplet delivery process.
(6) the nature of the impaction surface.

It would further be expected that each of these contributing factors would be influenced by a great number of elemental causes. Hence, it appears that a most promising means of an adequate description of the impaction process may, at least for the present, be via a random theory approach. The preceding statements tacitly lead up to the possibility that our theoretical considerations herein may be extendable beyond the transducer analysis to "real impaction processes." This is not to imply that a linear model will necessarily be the correct answer but perhaps only a start. Stochastic process concepts may offer possibilities of forming generalizations about droplet impaction processes and energy transfer which might not otherwise be apparent. Consequently, this mode of approach is suggested as a possible line of future research.

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